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Engineering Notes

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Nonfree Space Radiating Fin Optimum Dimension and Efficiency Correlations

C. K. Krishnaprakas* and K. Badari Narayana†
ISRO Satellite Center, Bangalore 560 017, India

Introduction

RADIATING fins find application in spacecraft thermal control for rejecting away the internally dissipated heat in various electronic components onboard to cold space. A number of references are available in literature on radiating fins.[1–3] The results are often expressed in the form of charts and require interpolation and cross plotting in order to use them. This is cumbersome, and hence they are not convenient in computer-based engineering design and analysis practice. Chang[4] presented a simple correlation for the fin efficiency, but is limited to the case of a fin radiating to a free space. Quite often, situations like the radiating fin receiving thermal loads, such as solar, earthshine, and albedo (reflected solar load from the Earth) from the space environment as experienced in a low Earth orbit and other external loads (both reflected and emitted) from the other spacecraft appendages, are encountered. In such cases the fin is, in effect, radiating to a nonfree space temperature determined by the sum of all external heating loads. There is a need for a suitable formula to estimate quickly the heat radiated out by a fin of a given length, thickness, emittance of the surface, base temperature, and the ambient temperature. A correlation is proposed in this Note to estimate the efficiency of a longitudinal rectangular fin radiating to a nonfree space. This correlation is applicable to a wide range of values of the so-called parameter, radiation-conduction number N_r. The correlation is useful in estimating the heat removal capacity of radiating fins for spacecraft applications. Another correlation to determine the optimum N_r, yielding the least fin material for different background temperatures also, is presented. The results of the optimum design are compared with results by previous investigators.

Analysis

Consideration is given to a one-dimensional longitudinal rectangular fin of length L (meters) and thickness t (meters), radiating to space and receiving the environmental loads such as the solar load, the reflected solar load (albedo), and the planetary thermal loads. The thermal conductivity of the fin material and the emittance of the surface are not temperature dependent. Also the fin radiates from both sides but receives solar, albedo, and earthshine loads from only one side. The governing differential equation under steady-state conditions is written as

$$kt\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = 2\varepsilon\sigma\left(T^4 - T_s^4\right) - \alpha(I_{\mathrm{sc}}\cos\varphi + q_{\mathrm{alb}}) - \varepsilon q_{\mathrm{earth}} \qquad (1)$$

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*Engineer, Thermal Systems Group.
†Head, Thermal Design Section, Thermal Systems Group.

In the preceding equation, α and ε are, respectively, the solar absorptance and infrared emittance of the fin surface, σ is the Stephan–Boltzmann constant (5.669×10^{-8} W/m^2-K^4), φ is the solar incidence angle measured from the fin surface normal, k is the thermal conductivity of the fin material (W/m-K), I_{sc} is the solar constant (1353 W/m^2), q_{alb} is the albedo load (W/m^2), q_{earth} is the earthshine load (W/m^2), T is the fin temperature (K), T_s is the space temperature (4 K), and x is the x coordinate of a point on the fin. It is convenient to define an effective background temperature of the surroundings caused by radiative heating as

$$T_\infty = \left\{\left(T_s^4\right) + \frac{[\alpha(I_{\mathrm{sc}}\cos\varphi + q_{\mathrm{alb}}) + \varepsilon q_{\mathrm{earth}}]}{2\varepsilon\sigma}\right\}^{\frac{1}{4}} \qquad (2)$$

T_∞ is similar to the definition of the so-called sol–air temperature used in air-conditioning and solar energy engineering practice, but takes into consideration here radiation heat transfer instead of convection heat transfer.

We can rewrite Eq. (1) using T_∞ as

$$kt\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = \varepsilon\sigma\left(T^4 - T_\infty^4\right) \qquad (3)$$

The following dimensionless parameters are now introduced:

$$\theta = \frac{T}{T_b}, \qquad \xi = \frac{x}{L}, \qquad N_r = \frac{2\varepsilon\sigma T_b^3 L^2}{kt}, \qquad \theta_\infty = \frac{T_\infty}{T_b}$$

T_b is the temperature at the base ($x = 0$) of the fin, and θ and ξ are respectively the dimensionless temperature and distance. N_r is called the radiation-to-conduction number that measures the dominance of radiation over conduction. Using the preceding dimensionless numbers, Eq. (3) can be written in dimensionless form as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = N_r\left(\theta^4 - \theta_\infty^4\right) \qquad (4)$$

The fin tip is insulated so that the heat transfer is zero there, that is, $x = L$, $\mathrm{d}T/\mathrm{d}x = 0$, or $\xi = 1$, $\mathrm{d}\theta/\mathrm{d}\xi = 0$. The boundary conditions for Eq. (4) are

$$\xi = 0, \qquad \theta = 1, \qquad \text{and} \qquad \xi = 1, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\xi} = 0 \qquad (5)$$

An analytical solution of the nonlinear second-order differential equation, Eq. (4), along with the boundary condition, Eq. (5), seems impossible; however, the equation can be reduced to a first-order one. Integrating Eq. (4) between the limits $\xi = \xi$ to $\xi = 1$, using the identity $\mathrm{d}^2\theta/\mathrm{d}\xi^2 = [\mathrm{d}[(\mathrm{d}\theta/\mathrm{d}\xi)^2]/\mathrm{d}\xi]/(2\mathrm{d}\theta/\mathrm{d}\xi)$, and after certain algebraic manipulations leads to[1]

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\sqrt{\frac{2}{5}N_r\left[\left(\theta^5 - \theta_t^5\right) - 5\theta_\infty^4(\theta - \theta_t)\right]} \qquad (6)$$

where $\theta_t = T_t/T_b$ and T_t is the temperature at the fin tip. The negative sign in Eq. (6) is taken because the temperature gradient is negative in view that heat flows from base to tip. The heat flux q at the base per unit width is

$$q = -kt\frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=0}, \qquad \text{or} \qquad -\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\bigg|_{\xi=0} = \frac{qL}{ktT_b} = \Phi \tag{7}$$

where $\Phi = qL/(ktT_b)$ is the dimensionless heat dissipated by the fin to the surroundings. Inserting the boundary condition, $\xi = 0$, $\theta = 1$ in Eq. (6), and using the resulting expression for $\mathrm{d}\theta/\mathrm{d}\xi$ in Eq. (7), we get

$$\Phi = \sqrt{\tfrac{2}{5}N_r\left[\left(1-\theta_t^5\right) - 5\theta_\infty^4(1-\theta_t)\right]} \tag{8}$$

Equation (8) expresses the heat transferred by the fin to the surroundings in terms of the radiation-to-conduction number, the tip temperature, and the background temperature. The effectiveness of the fin is generally expressed in terms of fin efficiency defined as the ratio of the heat transferred by the fin to the surroundings to that heat transferred by the fin when the entire fin is maintained at the same temperature as the base temperature, that is,

$$\eta = \frac{q}{2L\varepsilon\sigma\left(T_b^4 - T_\infty^4\right)} = \frac{-kt(\mathrm{d}T/\mathrm{d}x)\big|_{x=0}}{2L\varepsilon\sigma\left(T_b^4 - T_\infty^4\right)}$$

$$= -\frac{kt}{2\varepsilon\sigma T_b^3L^2}\frac{(\mathrm{d}\theta/\mathrm{d}\xi)\big|_{\xi=0}}{\left(1-\theta_\infty^4\right)} \tag{9}$$

and using Eq. (7) in Eq. (9) results in

$$\eta = (1/N_r)\left[\Phi\big/\left(1-\theta_\infty^4\right)\right] \tag{10}$$

Equation (10) implies that the efficiency of the radiating fin is a function of two parameters N_r and θ_∞, that is, $\eta = \eta(N_r, \theta_\infty)$. Efficiency increases with lower values of N_r and θ_∞. For the evaluation fin efficiency, Eq. (10) needs the values of N_r, θ_∞, and $\Phi = \Phi(N_r, \theta_\infty)$. Given N_r and θ_∞, θ_t is determined such that when Eq. (6) is integrated to the endpoint $\xi = 1$ satisfies the condition $\mathrm{d}\theta/\mathrm{d}\xi = 0$. Φ is then evaluated using Eq. (8).

Optimum Design

An optimum design of fin is required to keep the weight to a minimum. For a given profile area $A = Lt$, we can write $N_r = 2\varepsilon\sigma T_b^3L^2/kt = 2\varepsilon\sigma T_b^3L^3/kA$, that is, $L \propto N_r^{1/3}$. Therefore, for the condition of constant A, k, ε, T_b, and T_∞, the heat transferred by the fin is found out from $q = \eta L2\varepsilon\sigma(T_b^4 - T_\infty^4)$ as

$$q = C\eta N_r^{\frac{1}{3}} \tag{11}$$

where C is a constant (Ref. 2, pp. 36–39). This shows that the optimum radiation-to-conduction number $N_{r_{\text{opt}}}$ that gives maximum heat transfer is the solution of the following maximization problem:

$$\text{Maximize} \qquad f(N_r) = \eta N_r^{\frac{1}{3}}, \qquad \text{for given} \qquad \theta_\infty \tag{12}$$

Numerical Scheme

The nonlinear first-order differential equation (6) along with the initial condition $\xi = 0$, $\theta = 1$ can be solved only if θ_t is known. The problem is solved by the shooting method.[5] The aim is to determine the correct value of θ_t such that when it is used in Eq. (10) and integrated yields $\theta_t^* = \theta_t$, where θ_t^* is the estimated solution of the fin governing differential equation, Eq. (6), at the fin tip $\xi = 1$. The problem now becomes that of determining the root of an equation in which the variable θ_t appears implicitly. A residual function can be defined as

$$f(\theta_t) = \theta_t^* - \theta_t \tag{13}$$

The root of the implicit equation (13) is calculated using Newton's method,[6] which is quadratically convergent. This method requires the solution of the ordinary differential equation (6) several times, as many times as the total number of iterations. An accurate eighth-order Runge–Kutta method from Cooper and Verner is employed for this purpose.[7] The optimum radiation-conduction number $N_{r_{\text{opt}}}$ is evaluated by solving Eq. (12) numerically using Powell's quadratic search method for finding the minimum point of a univariable function.[8]

Results

The convergence characteristics of the numerical method are studied for several cases. It is seen that approximately 50 uniform nodal divisions (number of steps for numerical integration) make the solution almost grid independent and give accurate numerical results. The efficiency for the case of a fin radiating to a free space, evaluated numerically is presented in Fig. 1. The results obtained using one of the earlier correlations from Chang[4] also are presented in Fig. 1. Chang's correlation for the fin efficiency is

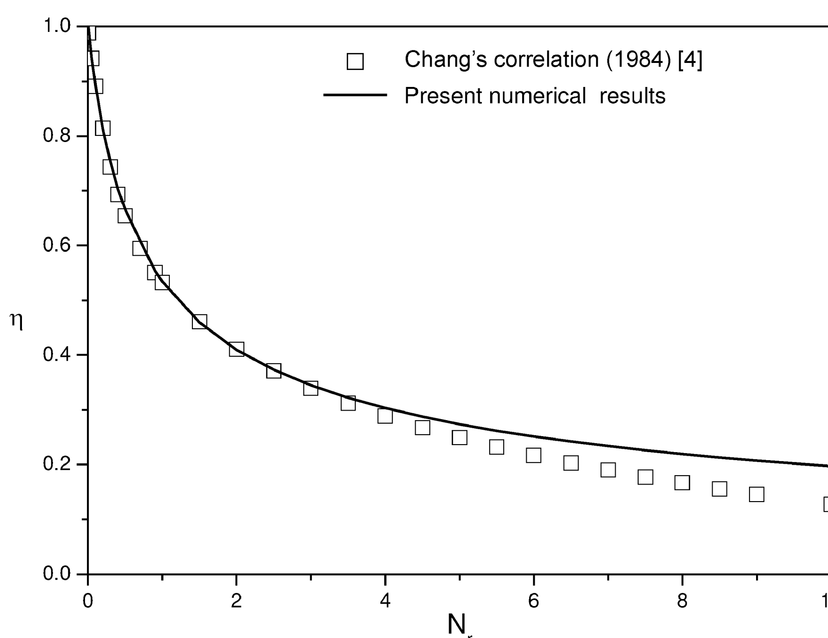

Fig. 1 Efficiency of a longitudinal rectangular fin radiating to free space ($\theta_\infty = 0$).

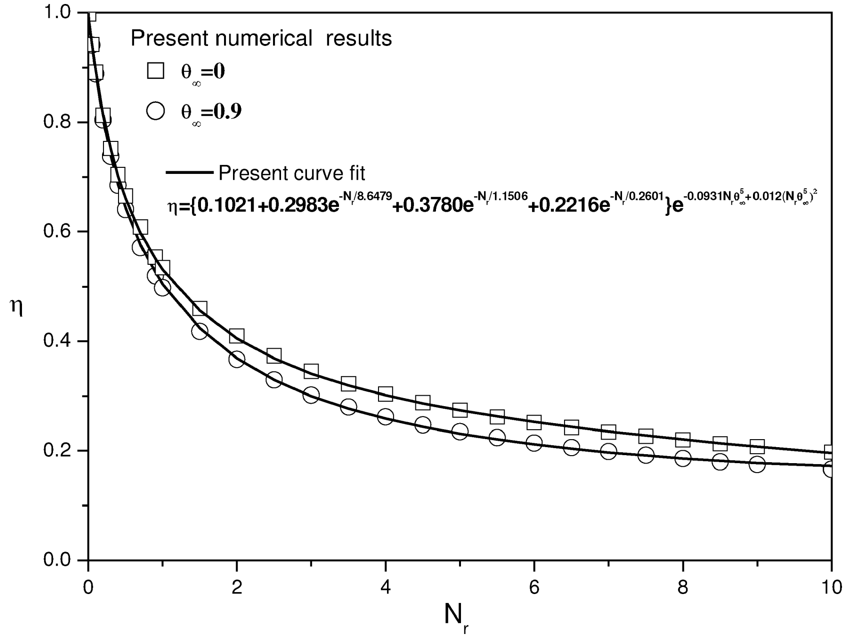

Fig. 2 Efficiency of a longitudinal rectangular fin radiating to a nonfree space.

$$\bar{\eta} = \begin{cases} \left(1 - 1.25N_r + 1.6N_r^2\right)\left(1 - \theta_\infty^4\right), & 0.01 \le N_r \le 0.2 \\ (-0.4049\log_{10} N_r + 0.5321)\left(1 - \theta_\infty^4\right), & 0.2 \le N_r \le 2.0 \end{cases} \tag{14}$$

where the fin efficiency $\bar{\eta}$ is defined as $\bar{\eta} = q/(L2\varepsilon\sigma T_b^4)$. The fin efficiency defined by Eq. (9) η can be obtained from $\bar{\eta}$ from the relation $\eta = \bar{\eta}/(1 - \theta_\infty^4)$. Therefore, we see that while using Chang's correlation η becomes a function of only a function of N_r because the term $(1 - \theta_\infty^4)$ is canceled out, and θ_∞ no longer appears. Or, in other words, the preceding correlation is applicable only for the case of a fin radiating to a free space. In Ref. 9 [p. 218, Eq. (6.5) for Chang's correlation], the coefficient for N_r is incorrectly typed as 1.125; this should be corrected to 1.25.

The fin efficiency calculated using the present numerical scheme is found to be in very good agreement with that of Chang for the case $\theta_\infty = 0$ and within the range of validity of N_r specified in Eq. (14). Beyond $N_r = 2$, Chang's correlation deviates from the present numerical results. In a spacecraft problem, the fin is radiating to an effective nonfree space in view of the environmental heating loads, and it is essential to evaluate fin efficiency for this condition. The present numerical results obtained using the shooting method are compared to the results obtained using an altogether different numerical technique called the boundary-element method,[10] and it is found that both the results agree very well, thus confirming the validity of the numerical results.

It is convenient to have a correlation to evaluate the fin efficiency so that the solution of Eq. (6) is not necessary for every case. A correlation obtained by the method of nonlinear least-squares curve fitting[8] of the numerical results is proposed now for a longitudinal rectangular fin radiating to a nonfree space as

$$\eta = (0.1021 + 0.2983e^{-N_r/8.6479} + 0.3780e^{-N_r/1.1506} + 0.2216e^{-N_r/0.2601})\exp\left[-0.0931N_r\theta_\infty^5 + 0.0120\left(N_r\theta_\infty^5\right)^2\right] \tag{15}$$

When $\theta_\infty = 0$, the preceding expression reduces to that for a free-space condition. The results obtained using the present correlation are in very good agreement with the numerical results as shown in Fig. 2, and the coefficient of correlation is very close to unity, thus proving the correlation.

Though the correlation is derived for the case where the fin is radiating from both sides, it can be easily extended for the case when only one side is radiating by replacing the value of the emittance term 2ε by ε in the definitions of N_r and T_∞. Once η is calculated, Φ can be evaluated using Eq. (10), and the tip temperature can be obtained by solving the nonlinear equation Eq. (8) for θ_t. Squaring Eq. (8) on both sides and rearranging leads to the nonlinear equation in θ_t as

$$f(\theta_t) = \tfrac{2}{5}N_r\left[\left(1 - \theta_t^5\right) - 5\theta_\infty^4(1 - \theta_t)\right] - \Phi^2 = 0 \tag{16}$$

The preceding equation can be solved by an iterative method such as Newton's method[6] or a polynomial root finding scheme such as the cubically convergent Laguerre's method.[11] Newton's method converges fast especially when the starting value is close to the root. A good starting value for θ_t can be taken as $\theta_t = \sqrt[5]{[1 - 5\Phi^2/(2N_r)]}$.

One might feel looking at Fig. 2 that the difference between the results for free-space and nonfree-space conditions is small so that the results for the latter can be approximated by the former, and hence the simple Chang's correlation can be used as done by some researchers.[12] But, this is likely to produce numerical instabilities in case the fin tip temperature equation (16) is numerically solved using a real root finder, say, Newton's method. This is studied for the case of $N_r = 1$ and $\theta_\infty = 0.9$. When the fin efficiency is evaluated for the actual θ_∞ condition, Newton's scheme converged very fast, in five iterations, whereas when the fin efficiency is approximated to that of $\theta_\infty = 0$ condition the numerical scheme does not behave well. Numerical oscillations start to build up, and iterations continue indefinitely. When Laguerre's method is employed to determine all of the roots of Eq. (16), a negative root and two complex conjugate pairs resulted. This explains the anomalous behavior of Newton's method because the nearest root in modulus lies in the imaginary plane the method goes oscillating. This is not anticipated by the Newton solver, which is written in the present case exclusively to deal with real roots. The situation is confusing, and it solely happened because the equation became inconsistent when Φ evaluated using η for $\theta_\infty = 0$ condition was used rather than that for the nonzero θ_∞ condition.

Optimum radiation-conduction number $N_{r_{\rm opt}}$ is evaluated by solving Eq. (12) numerically for different values of θ_∞, using the correlation Eq. (15) for $\eta(N_r, \theta_\infty)$ and is plotted in Fig. 3. The same figure shows also the optimum efficiency $\eta_{\rm opt}$ at various values of $N_{r_{\rm opt}}$. It is seen that $N_{r_{\rm opt}}$ decreases and $\eta_{\rm opt}$ increases as θ_∞ increases. A cubic functional fit in the independent variable θ_∞^4 reasonably matches with the numerical results of $N_{r_{\rm opt}}$ is shown in Fig. 3. The functional

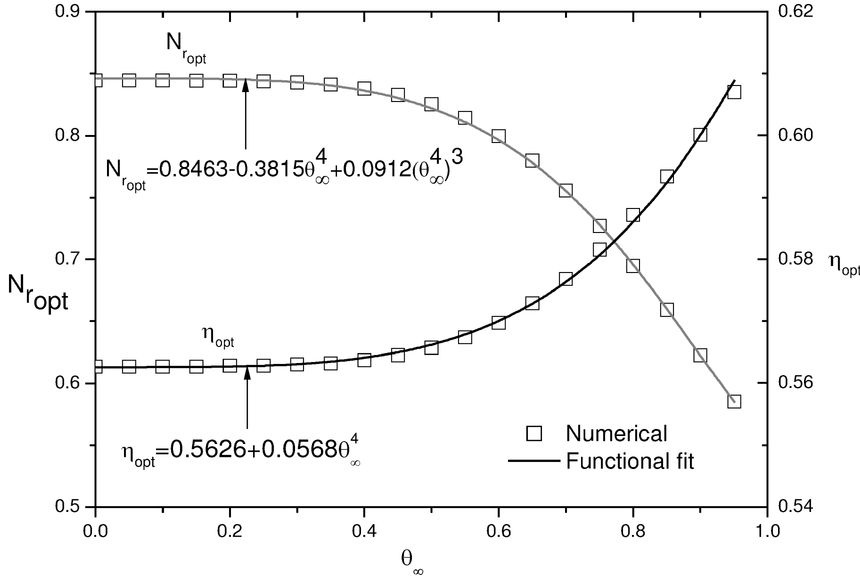

Fig. 3 Optimum radiation-to-conduction number and efficiency.

Table 1 Comparison of optimization analysis results

Investigator(s)	Year	Method	$N_{r_{\rm opt}}$	$\eta_{\rm opt}$
Liu[13,14]	1960, 1961	Lagrange multipliers	0.8039	0.579
Wilkins[15]	1960	Calculus of variations	0.8463	——
Chung and Nguyen[16]	1986	Numerical	0.8468	0.565
Smith[17]	1992	Cascade algorithm	0.8071	——
Present	2004	Numerical	0.8463	0.5627

fit has the form

$$N_{r_{\rm opt}} = 0.8463 - 0.3815\theta_\infty^4 + 0.0912\left(\theta_\infty^4\right)^3 \tag{17}$$

$\eta_{\rm opt}$ can be approximated as a linear functional fit in θ_∞^4 given by

$$\eta_{\rm opt} = 0.5626 + 0.0568\theta_\infty^4 \tag{18}$$

The results of the optimization analysis are verified by comparing the value of $N_{r_{\rm opt}}$ against the results presented in literature (Ref. 3, pp. 640–644) for the case of the fin radiating to a free space ($\theta_\infty = 0$). The results of the different previous investigators[13–17] vary, as presented in Table 1.

$N_{r_{\rm opt}}$ was estimated using Eqs. (14–26), (30), and (31) presented in Ref. 3 (p. 642) for the evaluation of optimum length and thickness. Discrepancies between the $N_{r_{\rm opt}}$ results by the previous investigators might be noticed. The present results agree well with those of Wilkins[15] and Chung and Nguyen.[16] The deviation between the $N_{r_{\rm opt}}$ result of Wilkins and the present result is only 0.2%. Incidentally, the present functional fit, Eq. (18), gives $N_{r_{\rm opt}} = 0.8463$, exactly equal to that reported by Wilkins. The results of Liu[13,14] and Smith[17] differ by 0.4%. Once $N_{r_{\rm opt}}$ is known, $\eta_{\rm opt}$ can be found out using Eq. (15). $L_{\rm opt}$ can be found out as $L_{\rm opt} = q/[\eta_{\rm opt} 2\varepsilon\sigma(T_b^4 - T_\infty^4)]$ and $t_{\rm opt}$ can then be determined from the definition of $N_{r_{\rm opt}}$ as $t_{\rm opt} = 2\varepsilon\sigma T_b^3 L_{\rm opt}^2 / (kN_{r_{\rm opt}})$.

Conclusions

Correlation (15) is suggested to evaluate the efficiency of a longitudinal rectangular fin radiating to a nonfree space. This correlation covers a wide range of the values of the parameter radiation-conduction number N_r, in comparison to Chang's correlation, and is useful in estimating the heat transferred by the fin to the surroundings without the need of solving a nonlinear second-order differential equation. The fin efficiency evaluated using Chang's correlation when used to find out the fin tip temperature for the case of nonfree-space condition led to meaningless results. This emphasizes the need to use the value of fin efficiency for the correct θ_∞ condition and should not be approximated by that at $\theta_\infty = 0$. A correlation, Eq. (17), to estimate the optimum N_r yielding the least fin material for different background temperatures also is given. Optimum N_r decreases with increasing background temperature.

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P. Huseman
Associate Editor